

Epimorphic subgroups of algebraic groups

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Abstract

In this note, we show that the epimorphic subgroups of an algebraic group are exactly the pull-backs of the epimorphic subgroups of its affinization. We also obtain epimorphicity criteria for subgroups of affine algebraic groups, which generalize a result of Bien and Borel. Moreover, we extend the affinization theorem for algebraic groups to homogeneous spaces.

1 Introduction and statement of the results

The algebraic groups considered in this note are the group schemes of finite type over a field k . They form the objects of a category, with morphisms being the homomorphisms of k -group schemes. One of the most basic questions one may ask about this category is to describe *monomorphisms* and *epimorphisms*. Recall that a morphism $f : G \rightarrow H$ is a monomorphism if it satisfies the left cancellation property: for any algebraic group G' and for any morphisms $f_1, f_2 : G' \rightarrow G$ such that $f \circ f_1 = f \circ f_2$, we have $f_1 = f_2$. Likewise, f is an epimorphism if it satisfies the right cancellation property.

The answer to this question is very easy and well-known for monomorphisms: these are exactly the homomorphisms with trivial (scheme-theoretic) kernel, or equivalently the closed immersions of algebraic groups. Also, $f : G \rightarrow H$ is an epimorphism if and only if so is the inclusion of its scheme-theoretic image. This reduces the description of epimorphisms to that of the *epimorphic subgroups* of an algebraic group G , i.e., of those algebraic subgroups H such that every morphism $G \rightarrow G'$ is uniquely determined by its pull-back to H . The purpose of this note is to characterize such subgroups.

Examples of epimorphic subgroups include the parabolic subgroups of a smooth connected affine algebraic group G , i.e., the algebraic subgroups $H \subset G$ such that the homogeneous space G/H is proper, or equivalently projective. Indeed, for any morphisms $f_1, f_2 : G \rightarrow G'$ which coincide on H , the map $G \rightarrow G', x \mapsto f_1(x) f_2(x^{-1})$ factors through a map $\varphi : G/H \rightarrow G'$. But every such morphism is constant: to see this, we may assume k algebraically closed; then G/H is covered by rational curves (images of morphisms $\mathbb{P}^1 \rightarrow G/H$), while every morphism $\mathbb{P}^1 \rightarrow G'$ is constant.

In the category of smooth connected affine algebraic groups over an algebraically closed field, the epimorphic subgroups have been studied by Bien and Borel in [BB92a, BB92b]; see also [Gr97, §23] for a more detailed exposition, and [BBK96, §4] for further developments. In particular, [BB92a, Thm. 1] presents several epimorphicity criteria in that setting. Our first result extends most of these criteria to affine algebraic groups over an arbitrary field. To state it, let us define *affine epimorphic subgroups* of an affine

algebraic group G as those algebraic subgroups $H \subset G$ that are epimorphic in the category of affine algebraic groups. (Clearly, epimorphic implies affine epimorphic. In fact, the converse holds, as we will show in Corollary 3).

Theorem 1. *The following conditions are equivalent for an algebraic subgroup H of an affine algebraic group G :*

- (i) H is affine epimorphic in G .
- (ii) $\mathcal{O}(G/H) = k$.
- (iii) For any finite-dimensional G -module V , we have the equality of fixed point subschemes $V^H = V^G$.
- (iv) For any finite-dimensional G -module V , if $V = V_1 \oplus V_2$, where V_1, V_2 are H -submodules, then V_1, V_2 are G -submodules.

This result is proved in Section 2 by adapting the argument of [BB92a, Thm. 1] (see also [ND05, Thm. 13]). When G is smooth and connected, condition (ii) is equivalent to the k -vector space $\mathcal{O}(G/H)$ being finite-dimensional. But this fails for non-connected groups (just take H to be the trivial subgroup of a non-trivial finite group G) and for non-smooth groups as well (take G, H as above with G infinitesimal).

For the category of finite-dimensional Lie algebras over a field of characteristic 0, the equivalence of conditions (i), (iii) and (iv) has been obtained by Bergman in an unpublished manuscript which was the starting point of [BB92a]; see [Be70, Cor. 3.2], and [Pa14] for recent developments based on the (related but not identical) notion of wide subalgebra of a semi-simple Lie algebra.

Our second result yields an epimorphicity criterion in the category of all algebraic groups. To formulate it, recall the affinization theorem (see [DG70, §III.3.8]): every algebraic group G has a smallest normal algebraic subgroup N such that the quotient G/N is affine. Moreover, N is smooth, connected, and contained in the center of the neutral component G^0 . Also, N is anti-affine (i.e., $\mathcal{O}(N) = k$) and N is the largest algebraic subgroup of G satisfying this property; we denote N by G_{ant} . The quotient morphism $G \rightarrow G/G_{\text{ant}}$ is the affinization morphism, i.e., the canonical map

$$\varphi_G : G \longrightarrow \text{Spec } \mathcal{O}(G).$$

We may now state:

Theorem 2. *The following conditions are equivalent for an algebraic subgroup H of an algebraic group G :*

- (i) H is epimorphic in G .
- (ii) $H \supset G_{\text{ant}}$ and $\mathcal{O}(G/H) = k$.

This result is proved in Section 4, after gathering auxiliary results in Section 3.

Note that the formations of $\mathcal{O}(G/H)$ and G_{ant} commute with base change by field extensions of k . In view of Theorem 2, this yields the first assertion of the following:

Corollary 3. *Let G be an algebraic group, and H an algebraic subgroup.*

- (i) *H is epimorphic in G if and only if the base change $H_{k'}$ is epimorphic in $G_{k'}$ for some field extension k' of k .*
- (ii) *When G is affine, H is epimorphic in G if and only if it is affine epimorphic.*

The second assertion follows readily from Theorems 1 and 2. As a consequence, *the epimorphic subgroups of an algebraic group G are exactly the pull-backs of the epimorphic subgroups of its affinization.*

Theorem 2 and Corollary 3 reduce the description of the epimorphic subgroups of an algebraic group G over k , to the case where G is affine and k is algebraically closed. When G is smooth, our next result yields further reductions:

Theorem 4. *The following conditions are equivalent for an algebraic subgroup H of a smooth algebraic group G over an algebraically closed field k :*

- (i) *H is epimorphic in G .*
- (ii) *The reduced subgroup H_{red} is epimorphic in G .*
- (iii) *The reduced neutral component H_{red}^0 is epimorphic in G^0 and the natural map $H_{\text{red}}/H_{\text{red}}^0 \rightarrow G/G^0$ is surjective.*

This result is proved in Section 5.

In view of the above results, the class of homogeneous spaces $X = G/H$ such that $\mathcal{O}(X) = k$ deserves further consideration. These anti-affine homogeneous spaces feature in an extension of the affinization theorem for algebraic groups (see [DG70, III.3.8]), which is our final result. To state it, recall that a quasi-compact scheme Z is said to be quasi-affine if the affinization map $\varphi_Z : Z \rightarrow \text{Spec } \mathcal{O}(Z)$ is an open immersion (see [EGA, II.5.1.2] for further characterizations).

Theorem 5. *Let G be an algebraic group, and H an algebraic subgroup.*

- (i) *There exists a smallest algebraic subgroup L of G containing H such that G/L is quasi-affine. Moreover, $\mathcal{O}(G/L) = \mathcal{O}(G/H)$ and the affinization map*

$$\varphi_{G/H} : G/H \longrightarrow \text{Spec } \mathcal{O}(G/H) =: X$$

is the composition

$$G/H \xrightarrow{u} G/L \xrightarrow{\varphi_{G/L}} \text{Spec } \mathcal{O}(G/L) = X,$$

where u denotes the canonical morphism.

- (ii) *The formation of L commutes with base change by arbitrary field extensions.*
- (iii) *L is the largest subgroup of G containing H such that L/H is anti-affine.*

- (iv) L/H is geometrically irreducible.
- (v) If G is affine, then L is the largest subgroup of G containing H as an epimorphic subgroup.
- (vi) If G and H are smooth, then so is L .

This result is proved in Section 6. In the setting of smooth affine algebraic groups over an algebraically closed field, it gives back a statement of Bien and Borel (see [BB92a, Prop. 1]), proved by Grosshans in [Gr97, §2, §23]; our proof, based on a descent argument, is somewhat more direct.

Also, Theorem 5 gives back most of the affinization theorem for an arbitrary algebraic group G . More specifically, taking for H the trivial subgroup and using the fact that every quasi-affine algebraic group is affine (see e.g. [SGA3, VIB.11.11]), we obtain that G has a smallest algebraic subgroup L such that G/L is affine, and L is the largest anti-affine subgroup of G ; moreover, L is connected. But the smoothness property of anti-affine algebraic groups does not extend to homogeneous spaces, as shown by Example 11 at the end of Section 6.

Returning to the description of all the epimorphic subgroups H of a smooth algebraic group G over a field k , we may assume (by Theorem 2, Corollary 3 and Theorem 4) G to be affine and connected, H smooth and connected, and k algebraically closed; this is precisely the setting of [BB92a, BB92b]. Even so, the structure of epimorphic subgroups is only partially understood; a geometric criterion of epimorphicity is obtained in [Pe11] when G is semi-simple and k has characteristic 0.

The classification of epimorphic subgroups of non-smooth algebraic groups presents further open problems; see Example 11 again for a construction of such subgroups $H \subset G$, for which the quotient G/H is non-smooth as well. Examples with a smooth quotient may be obtained as follows: over any algebraically closed field k of prime characteristic, there exist rational homogeneous projective varieties X such that the automorphism group scheme Aut_X is non-smooth (see [BSU13, Prop. 4.3.4]). Let then G denote the neutral component of Aut_X , and H the stabilizer of a k -rational point $x \in X$. Then G is affine, non-smooth, and $X \cong G/H$; as a consequence, H is non-smooth as well. Also, H is epimorphic in G , by Theorem 2 or a direct argument as for parabolic subgroups. Thus, the description of epimorphic subgroups of possibly non-smooth algebraic groups entails that of automorphism group schemes of rational homogeneous projective varieties, which seems to be completely unexplored.

Notation and conventions. We use the books [DG70] and [SGA3] as general references, and the expository text [Br15] for some further results.

Throughout this note, we consider schemes over a fixed field k . By an *algebraic group*, we mean a group scheme G of finite type over k ; we denote by $e = e_G \in G(k)$ the neutral element, and by G^0 the neutral component of G . The group law of G will be denoted multiplicatively: $(x, y) \mapsto xy$.

By a *subgroup* of G , we mean a k -subgroup scheme H ; then H is closed in G . *Morphisms* of algebraic groups are understood to be homomorphisms of k -group schemes.

Given a subgroup $H \subset G$ and a normal subgroup $N \triangleleft G$, we denote by $N \rtimes H$ the corresponding semi-direct product, and by $N \cdot H$ the scheme-theoretic image of the

morphism

$$N \rtimes H \longrightarrow G, \quad (x, y) \longmapsto xy.$$

Then $N \cdot H$ is a subgroup of G , and the natural map $H/N \cap H \rightarrow G/N$ is a closed immersion with image $N \cdot H/N$ (see e.g. [SGA3, VIIA.5.3.3]).

2 Proof of Theorem 1

(i) \Rightarrow (ii): The action of G on itself by right multiplication yields a G -module structure on the algebra $\mathcal{O}(G)$ (see [DG70, Ex. II.2.1.2]). Moreover, for any subgroup $K \subset G$ acting on $\mathcal{O}(G)$ by right multiplication, the natural map $\mathcal{O}(G/K) \rightarrow \mathcal{O}(G)^K$ is an isomorphism, as follows e.g. from [SGA3, Cor. VIA.3.3.3]. In particular, $\mathcal{O}(G/H) \cong \mathcal{O}(G)^H$ and $\mathcal{O}(G)^G = k$. Thus, it suffices to show that every $f \in \mathcal{O}(G)^H$ is fixed by G .

Consider the action of G on itself by left multiplication; this yields another G -module structure on $\mathcal{O}(G)$, and $\mathcal{O}(G)^H$ is a G -submodule. By [DG70, II.2.3.1], we may choose a finite-dimensional G -submodule $V \subset \mathcal{O}(G)^H$ that contains f . View V as a vector group (the spectrum of the symmetric algebra of the dual vector space) equipped with a compatible G -action, and consider the semi-direct product $G' := V \rtimes G$. Then G' is an affine algebraic group; moreover, the maps

$$f_1 : G \longrightarrow G', \quad g \longmapsto (0, g), \quad f_2 : G \longrightarrow G', \quad g \longmapsto (g \cdot f - f, g)$$

are two morphisms which coincide on H . Thus, $f_1 = f_2$, that is, f is fixed by G .

(ii) \Rightarrow (iii): Recall from [Ja03, I.3.2] that V^H is the subscheme of V associated with a linear subspace. So it suffices to show that every $v \in V^H(k)$ is fixed by G . Let $f \in \mathcal{O}(V)$. Then the assignment $g \mapsto f(g \cdot v)$ defines $f_v \in \mathcal{O}(G)^H \cong \mathcal{O}(G/H) = k$. Thus, we have $f_v(g) = f_v(e)$, that is, $f(g \cdot v) = f(v)$ identically on G . Since this holds for all $f \in \mathcal{O}(V)$, it follows that $g \cdot v = v$ identically on G .

(iii) \Rightarrow (iv): Let $\pi : V \rightarrow V_1$ denote the projection with kernel V_2 . Consider the action of G on $\text{End}(V)$ by conjugation; then $\text{End}(V)$ is a finite-dimensional G -module, and $\pi \in \text{End}(V)^H$. Thus, $\pi \in \text{End}(V)^G$. It follows that V_1 (the image of π) is normalized by G . Likewise, V_2 is normalized by G .

(iv) \Rightarrow (i): Let G' be an affine algebraic group, and

$$f_1, f_2 : G \longrightarrow G'$$

two morphisms which coincide on H . We may view G' as a subgroup of $\text{GL}(V)$ for some finite-dimensional vector space V (see [DG70, II.2.3.3]). This yields two linear representations

$$\rho_1, \rho_2 : G \longrightarrow \text{GL}(V)$$

which coincide on H . Consider the morphism

$$\rho_1 \oplus \rho_2 : G \longrightarrow \text{GL}(V \oplus V).$$

Then we have with an obvious notation:

$$V \oplus V = (V \oplus 0) \oplus \text{diag}(V),$$

where $V \oplus 0$ is normalized by G , and $\text{diag}(V)$ is normalized by H (as $\rho_1|_H = \rho_2|_H$). So $\text{diag}(V)$ is normalized by G , that is, $\rho_1 = \rho_2$. Thus, $f_1 = f_2$.

3 Some auxiliary results

Throughout this section, G denotes an algebraic group, and $H \subset G$ a subgroup. We begin with a series of easy observations.

Lemma 6. *Assume that H is epimorphic in G .*

- (i) *If $K \subset G$ is a subgroup containing H , then K is epimorphic in G .*
- (ii) *If $N \triangleleft G$ is a normal subgroup, then $H/N \cap H$ is epimorphic in G/N .*

Proof. The assertion (i) is obvious, and implies that $N \cdot H$ is epimorphic in G . As a direct consequence, $N \cdot H/N$ is epimorphic in G/N ; this yields the assertion (ii). \square

Lemma 7. *Let H be an epimorphic subgroup of G .*

- (i) *If G is finite, then $G = H$.*
- (ii) *For an arbitrary G , we have $G = G^0 \cdot H$.*

Proof. (i) Since G is affine and $H \subset G$ is affine epimorphic, we have $\mathcal{O}(G/H) = k$ by Theorem 1. As the scheme G/H is finite and contains a k -rational point x , it follows that this scheme consists of the point x , hence $H = G$.

(ii) By Lemma 6 (ii), $G^0 \cdot H/G^0$ is epimorphic in G/G^0 . Thus, we may replace G with G/G^0 , and hence assume that G is finite and étale. Then $H = G$ by (i). \square

Remark 8. (i) For finite étale groups, Lemma 7 (i) also follows by adapting the proof of the surjectivity of epimorphisms of abstract groups, given in [Li70].

(ii) Lemmas 6 and 7 also hold in the category of affine algebraic groups, with the same proofs.

Lemma 9. *Let $N \triangleleft G$ be a normal subgroup. If $H \supset G_{\text{ant}}$, then $H/N \cap H \supset (G/N)_{\text{ant}}$. Conversely, if $H/N \cap H \supset (G/N)_{\text{ant}}$ and N is affine, then $H \supset G_{\text{ant}}$.*

Proof. By [Br15, Lem. 3.3.6], the natural map $G_{\text{ant}}/N \cap G_{\text{ant}} \rightarrow (G/N)_{\text{ant}}$ is an isomorphism. This yields the first assertion.

Conversely, assume that $H/N \cap H \supset (G/N)_{\text{ant}}$; equivalently, $(H/N \cap H)_{\text{ant}} = (G/N)_{\text{ant}}$. Using [Br15, Lem. 3.3.6] again, it follows that $G_{\text{ant}} \subset N \cdot H_{\text{ant}}$. Thus, it suffices to show that $(N \cdot H)_{\text{ant}} = H_{\text{ant}}$. Using once more [Br15, Lem. 3.3.6], it suffices in turn to check that $(N \rtimes H)_{\text{ant}} = H_{\text{ant}}$. Since N is affine and $N \rtimes H \cong N \times H$ as schemes, the affinization morphism

$$\varphi_{N \rtimes H} : N \rtimes H \longrightarrow \text{Spec } \mathcal{O}(N \rtimes H)$$

is identified with

$$\text{id} \times \varphi_H : N \times H \longrightarrow N \times \text{Spec } \mathcal{O}(H).$$

Taking fibers at e yields the desired equality. \square

Next, we obtain a result of independent interest, which generalizes (and builds on) Lemma 7 (i):

Lemma 10. *If G is proper and H is epimorphic in G , then $H = G$.*

Proof. The largest anti-affine subgroup G_{ant} is smooth, connected and proper, that is, an abelian variety. Moreover, the quotient group G/G_{ant} is proper and affine, hence finite. Thus, using Lemma 6 (ii) and Lemma 7 (i), it suffices to show that H contains G_{ant} .

We now reduce to the case where G and H are smooth. For this, we may assume that k has prime characteristic p . Denote by G_n (resp. H_n) the kernel of the n th relative Frobenius morphism of G (resp. H). Then G_n and H_n are infinitesimal; also, G/G_n and H/H_n are smooth for $n \gg 0$ (see [SGA3, VIIA.8.3]). Using Lemma 6 (ii) again together with Lemma 9, we see that it suffices to show that $H/H_n = H/G_n \cap H$ contains $(G/G_n)_{\text{ant}}$. This yields the desired reduction.

Under this smoothness assumption, $G^0 = G_{\text{ant}}$ is an abelian variety. Also, we have $G = G^0 \cdot H$ by Lemma 7 (ii). Thus, $G^0 \cap H$ is centralized by G^0 and normalized by H , and hence is a normal subgroup of G . Using Lemma 6 (ii) again, we may replace G , resp. H with $G/G^0 \cap H$, resp. $H/G^0 \cap H$, and hence assume in addition that $G^0 \cap H$ is trivial.

Under these assumptions, we may identify G with $G^0 \rtimes H$. Consider the diagonal action of H on $G^0 \times G^0$ and form the semi-direct product $G' := (G^0 \times G^0) \rtimes H$. Then the maps

$$\begin{aligned} f_1 : G &\longrightarrow G', & (x, y) &\longmapsto (x, e, y), \\ f_2 : G &\longrightarrow G', & (x, y) &\longmapsto (x, x, y), \end{aligned}$$

are two morphisms which coincide on H . Thus, $f_1 = f_2$. But then G^0 must be trivial. \square

4 Proof of Theorem 2

(i) \Rightarrow (ii): By [Br15, Thm. 2], G has a smallest normal subgroup N such that G/N is proper; moreover, N is affine. If H is epimorphic in G , then $H/H \cap N$ is epimorphic in G/N by Lemma 6 (ii). Using Lemma 10, it follows that $H/H \cap N = G/N$. So $H \supset G_{\text{ant}}$ by Lemma 9. Thus, $\bar{H} := H/G_{\text{ant}}$ is epimorphic in $\bar{G} := G/G_{\text{ant}}$ by Lemma 6 (ii) again. In view of Theorem 1, this yields $\mathcal{O}(\bar{G}/\bar{H}) = k$. As

$$\mathcal{O}(\bar{G}/\bar{H}) \cong \mathcal{O}(\bar{G})^{\bar{H}} = \mathcal{O}(G/G_{\text{ant}})^H \cong \mathcal{O}(G/H),$$

we obtain $\mathcal{O}(G/H) = k$.

(ii) \Rightarrow (i): Let again $\bar{G} := G/G_{\text{ant}}$ and $\bar{H} := H/G_{\text{ant}}$. Then $\mathcal{O}(\bar{G}/\bar{H}) = k$ by the above argument. Using Theorem 1, it follows that \bar{H} is affine epimorphic in \bar{G} . Together with Lemma 7 (ii) and Remark 8 (ii), this yields $\bar{G} = \bar{G}^0 \cdot \bar{H}$, and hence $\mathcal{O}(\bar{G}/\bar{H}) \cong \mathcal{O}(\bar{G}^0/\bar{G}^0 \cap \bar{H})$. By Theorem 1 again, it follows that $\bar{G}^0 \cap \bar{H}$ is affine epimorphic in \bar{G}^0 . Also, note that $G = G^0 \cdot H$, since G_{ant} is connected and contained in H .

Let $f_1, f_2 : G \rightarrow G'$ be morphisms that coincide on H . Then f_1, f_2 pull back to morphisms $f_1^0, f_2^0 : G^0 \rightarrow G'^0$ which coincide on $G_{\text{ant}} \triangleleft G^0 \cap H$. Moreover, the common scheme-theoretic image of G_{ant} under f_1^0, f_2^0 is contained in $G'_{\text{ant}} \triangleleft G'^0$. This yields morphisms of affine algebraic groups

$$\bar{f}_1^0, \bar{f}_2^0 : \bar{G}^0 \rightarrow G'^0/G'_{\text{ant}}$$

which coincide on $\bar{G}^0 \cap \bar{H}$. Thus, $\bar{f}_1^0 = \bar{f}_2^0$, that is, the morphism of schemes

$$\varphi : G^0 \longrightarrow G'^0, \quad x \longmapsto f_1(x)f_2(x)^{-1}$$

factors through G'_{ant} . We have

$$\varphi(xy) = f_1(x)f_1(y)f_2(y)^{-1}f_2(x)^{-1}$$

identically on $G^0 \times G^0$. Since G'_{ant} is contained in the center of G'^0 , it follows that φ is a morphism of algebraic groups.

As f_1 and f_2 coincide on $G_{\text{ant}} \subset H$, the kernel of φ contains G_{ant} . Thus, φ factors through a morphism of algebraic groups $\psi : \bar{G}^0 \rightarrow G'_{\text{ant}}$. Since \bar{G}^0 is affine, so is the scheme-theoretic image of ψ . Also, ψ is trivial on $\bar{G}^0 \cap \bar{H}$, an affine epimorphic subgroup of \bar{G}^0 . Thus, ψ is trivial, that is, f_1 and f_2 coincide on G^0 . Since these morphisms also coincide on H , and $G = G^0 \cdot H$, we conclude that $f_1 = f_2$.

5 Proof of Theorem 4

(i) \Rightarrow (ii): Recall from Theorem 2 that $G_{\text{ant}} \subset H$ and $\mathcal{O}(G/H) = k$. Since G_{ant} is smooth, it is contained in H_{red} . Thus, using Theorem 2 again, it suffices to show that $\mathcal{O}(G/H_{\text{red}}) = k$.

The natural map $u : G/H_{\text{red}} \rightarrow G/H$ lies in a commutative square

$$\begin{array}{ccc} G \times H/H_{\text{red}} & \xrightarrow{p_1} & G \\ m \downarrow & & \downarrow q \\ G/H_{\text{red}} & \xrightarrow{u} & G/H, \end{array}$$

where p_1 denotes the projection, q the quotient map, and m the pull-back of the action map $G \times G/H_{\text{red}} \rightarrow G/H_{\text{red}}$. In fact, this square is cartesian and consists of faithfully flat morphisms (see e.g. the proof of [Br15, Prop. 2.8.4]). As the scheme H/H_{red} is finite and has a unique k -rational point, the map p_1 is finite and purely inseparable; thus, so is u by faithfully flat descent. Also, G/H_{red} and G/H are smooth, since so is G . Thus, the induced map on rings of rational functions

$$u^\# : k(G/H) \longrightarrow k(G/H_{\text{red}})$$

is injective, and there exists a positive integer n (a power of the characteristic exponent of k) such that

$$k(G/H_{\text{red}})^n \subset u^\# k(G/H).$$

Also, by normality of G/H , we have $u^\# \mathcal{O}(G/H) = u^\# k(G/H) \cap \mathcal{O}(G/H_{\text{red}})$ and hence

$$\mathcal{O}(G/H_{\text{red}})^n \subset u^\# \mathcal{O}(G/H).$$

Since $\mathcal{O}(G/H) = k$ and $\mathcal{O}(G/H_{\text{red}})$ has no non-zero nilpotents, this yields the desired assertion.

(ii) \Rightarrow (iii): We may replace H with H_{red} , and hence assume that H is smooth. By Lemma 7 (ii), we have $G = G^0 \cdot H$; thus, the natural map $H/H^0 \rightarrow G/G^0$ is surjective. Also, G_{ant} is connected, and contained in H by Theorem 2; hence $G_{\text{ant}} \subset H^0$. So, using Theorem 2 once more, we are reduced to checking that $\mathcal{O}(G^0/H^0) = k$.

Note that

$$k = \mathcal{O}(G/H) = \mathcal{O}(G^0 \cdot H/H) \cong \mathcal{O}(G^0/G^0 \cap H).$$

Next, consider the natural map

$$\psi : G^0/H^0 \longrightarrow G^0/G^0 \cap H.$$

The finite étale group $F := (G^0 \cap H)/H^0 \subset H/H^0$ acts on the right on G^0/H^0 , and ψ is the categorical quotient for that action. Thus, $\mathcal{O}(G^0/H^0)^F \cong \mathcal{O}(G^0/G^0 \cap H)$, and hence the algebra $\mathcal{O}(G^0/H^0)$ is integral over $\mathcal{O}(G^0/G^0 \cap H) = k$. As above, this implies the desired assertion.

(iii) \Rightarrow (i): This follows by reverting some of the previous arguments. More specifically, we have $G_{\text{ant}} = (G^0)_{\text{ant}} \subset H_{\text{red}}^0 \subset H$. Also, $G = G^0 \cdot H_{\text{red}} = G^0 \cdot H$ and hence

$$\mathcal{O}(G/H) \cong \mathcal{O}(G^0/G^0 \cap H) \cong \mathcal{O}(G^0)^{G^0 \cap H} \subset \mathcal{O}(G^0)^{H^0} = k.$$

Thus, H is epimorphic in G by Theorem 2 again.

6 Proof of Theorem 5

(i) Consider the action of G on $\mathcal{O}(G)$ via right multiplication and let $L \subset G$ be the centralizer of the subspace $\mathcal{O}(G)^H \subset \mathcal{O}(G)$. In view of [DG70, II.1.3.6], L is represented by a subgroup of G that we will also denote by L . Since L acts trivially on $\mathcal{O}(G)^H$, we have $\mathcal{O}(G)^H \subset \mathcal{O}(G)^L$. On the other hand, $H \subset L$ and hence $\mathcal{O}(G)^L \subset \mathcal{O}(G)^H$. Thus, $\mathcal{O}(G)^L = \mathcal{O}(G)^H$.

We show that there exists a finite subset $F \subset \mathcal{O}(G)^H$ such that L is the centralizer $C_G(F)$. Indeed, we may find F such that $C_G(F)$ is minimal among all such centralizers. Then $C_G(F \cup \{f\}) = C_G(F)$ for any $f \in \mathcal{O}(G)^H$, and hence $C_G(F)$ centralizes the whole subspace $\mathcal{O}(G)^H$.

Choose $F = \{f_1, \dots, f_n\} \subset \mathcal{O}(G)^H$ such that $L = C_G(F)$. Then L is the centralizer in G of $f_1 + \dots + f_n$, viewed as a k -rational point of $\mathcal{O}(G) \oplus \dots \oplus \mathcal{O}(G) =: n\mathcal{O}(G)$. As f_1, \dots, f_n are contained in some finite-dimensional G -submodule $V \subset n\mathcal{O}(G)$, it follows that G/L is isomorphic to a subscheme of the affine space associated with V (see [DG70, III.3.5.2]). In view of [EGA, II.5.1.2], it follows that G/L is quasi-affine. In other terms, the affinization map $\varphi_{G/L}$ is an open immersion. Since $\mathcal{O}(G/L) = \mathcal{O}(G/H)$, this yields the desired commutative triangle

$$\begin{array}{ccc} G/H & & \\ \downarrow u & \searrow \varphi_{G/H} & \\ G/L & \xrightarrow{\varphi_{G/L}} & X, \end{array}$$

where u denotes the natural map, and $X = \text{Spec } \mathcal{O}(G/H) = \text{Spec } \mathcal{O}(G/L)$.

Let K be a subgroup of G such that $K \supset H$ and G/K is quasi-affine. Then we have a commutative square of G -equivariant morphisms

$$\begin{array}{ccc} G/H & \xrightarrow{\varphi_{G/H}} & \operatorname{Spec} \mathcal{O}(G/H) = \operatorname{Spec} \mathcal{O}(G/L) \\ v \downarrow & & \downarrow \varphi_v \\ G/K & \xrightarrow{\varphi_{G/K}} & \operatorname{Spec} \mathcal{O}(G/K), \end{array}$$

where $\varphi_{G/K}$ is an open immersion. Thus, v factors through u , and hence $L \subset K$.

(ii) In view of [DG70, I.1.2.6], the formation of the affinization morphism commutes with arbitrary field extensions. Thus, so does the formation of L .

(iii) Consider a subgroup K of G containing H such that K/H is anti-affine. Denote by $q : G \rightarrow G/H$ the quotient map and by $x = q(e_G)$ the base point. Then the pull-back map $\mathcal{O}(G)^H \rightarrow \mathcal{O}(K)^H \cong \mathcal{O}(K/H) = k$ is identified with the homomorphism $\mathcal{O}(G/H) \rightarrow k$ given by evaluation at x . Thus, $K/H \subset G/H$ is contained in the fiber of $\varphi_{G/H}$ at x . By (i), this fiber is $L/H \subset G/H$. It follows that $K \subset L$.

We now show that L/H is anti-affine. As in the proof of Theorem 4, we have a cartesian diagram of faithfully flat morphisms

$$\begin{array}{ccc} G \times L/H & \xrightarrow{p_1} & G \\ m \downarrow & & \downarrow r \\ G/H & \xrightarrow{u} & G/L, \end{array}$$

where p_1 denotes the projection, r the quotient map, and m the pull-back of the action map $G \times G/H \rightarrow G/H$. Thus, we obtain a canonical isomorphism of sheaves on G :

$$r^*(u_* \mathcal{O}_{G/H}) \xrightarrow{\cong} (p_1)_*(m^* \mathcal{O}_{G/H}).$$

Clearly, we have $m^* \mathcal{O}_{G/H} = \mathcal{O}_{G \times L/H}$ and $r^* \mathcal{O}_{G/L} = \mathcal{O}_G$. Moreover, the natural map $\mathcal{O}_{G/L} \rightarrow u_* \mathcal{O}_{G/H}$ is an isomorphism, since $\mathcal{O}(G/L) = \mathcal{O}(G/H)$ and G/L is covered by open affine subschemes of the form $(G/L)_f$, where $f \in \mathcal{O}(G/L)$ (see [EGA, II.5.1.2]). It follows that the natural map $\mathcal{O}_G \rightarrow (p_1)_* \mathcal{O}_{G \times L/H}$ is an isomorphism as well. In particular, this yields $\mathcal{O}(G) = \mathcal{O}(G \times L/H)$, and hence $\mathcal{O}(L/H) = k$ as desired.

(iv) It suffices to show that the natural map $L^0/L^0 \cap H \rightarrow L/H$ is an isomorphism, as every homogeneous space under a connected algebraic group is geometrically irreducible (see e.g. [SGA3, VIA.2.6.6]). The quotient $L/L^0 \cdot H$ is finite and étale (since so is L/L^0), and anti-affine (since so is L/H). Thus, this quotient consists of a unique k -rational point. Hence $L = L^0 \cdot H$; this yields the desired assertion.

(v) Let K be a subgroup of G containing H . As K is affine, we have by Theorem 2 that K/H is anti-affine if and only if H is epimorphic in K . In view of (ii), this yields the assertion.

(vi) By (ii), we may assume that k is algebraically closed. Then $H \subset L_{\text{red}} \subset L$ and the natural map $G/L_{\text{red}} \rightarrow G/L$ is finite, as shown in the proof of Theorem 4. Since G/L is quasi-affine, so is G/L_{red} in view of [EGA, II.5.1.2, II.5.1.12]. Thus, $L = L_{\text{red}}$ by the minimality of L , i.e., L is smooth.

Example 11. Assume that k has characteristic $p > 0$. Let $Y = G/H$ be a smooth anti-affine homogeneous space, where G is affine and $H \subsetneq G$. We will construct a non-smooth anti-affine homogeneous space X under an algebraic group containing G , such that X contains Y as its largest smooth subscheme. For this, we use a process of “infinitesimal thickening” of an arbitrary homogeneous space G/H .

Let M be a finite-dimensional G -module. Viewing M as a p -Lie algebra with zero bracket and p th power map, we obtain a commutative infinitesimal algebraic group $G_p(M)$ of height 1 (see [SGA3, VIIA.8.1.2]). The action of G on M yields an action on $G_p(M)$ by automorphisms of algebraic groups; we denote by $G_p(M) \rtimes G$ the corresponding semi-direct product.

Next, let $N \subset M$ be an H -submodule. As above, we may form the semi-direct product $G_p(N) \rtimes H$; this is a subgroup of $G_p(M) \rtimes G$. Consider the homogeneous space

$$X := G_p(M) \rtimes G / G_p(N) \rtimes H.$$

The chain of inclusions $G_p(N) \rtimes H \subset G_p(N) \rtimes G \subset G_p(M) \rtimes G$ yields a morphism

$$f : X \longrightarrow G_p(M) \rtimes G / G_p(N) \rtimes H \cong G/H = Y.$$

Moreover, f is G -equivariant and its fiber at the base point $y \in Y(k)$ is H -equivariantly isomorphic to $G_p(M)/G_p(N)$. The latter quotient is canonically isomorphic to $G_p(M/N)$, by [SGA3, VIIA.8.1.3]. The neutral element of $G_p(M/N)$ is fixed by H , and hence yields a section $s : Y \rightarrow X$ of $f : X \rightarrow Y$. As $G_p(M/N)$ is infinitesimal, f and s induce mutually inverse homeomorphisms of the underlying topological spaces of X and Y .

We have an isomorphism

$$\mathcal{O}(X) \cong (\mathcal{O}(G) \otimes \mathcal{O}(G_p(M/N)))^H,$$

where H acts simultaneously on $\mathcal{O}(G)$ by left multiplication, and on $\mathcal{O}(G_p(M/N))$ via its linear action on M/N . Also, recall from [SGA3, VIIA.7.4] the canonical isomorphism

$$\mathcal{O}(G_p(M/N)) \cong \mathrm{Sym}(M/N)^*/I,$$

where $\mathrm{Sym}(M/N)^*$ denotes the symmetric algebra of the dual module of M/N , and I the ideal generated by the p th powers of all elements of $(M/N)^*$.

Assume that G is affine. By a theorem of Chevalley (see e.g. [DG70, II.2.3.5]), we may choose a finite-dimensional G -module M and a hyperplane $N \subset M$ such that H is the stabilizer of N for the G -action on M . In particular, N is an H -submodule of M ; we denote by $L = M/N$ the quotient line. Then we have an isomorphism of H -modules

$$\mathcal{O}(G_p(M/N)) \cong \bigoplus_{i=0}^{p-1} L^{-i},$$

where L^{-i} denotes the i th tensor power of L^* (in particular, L^0 is the trivial H -module k). Denoting by \mathcal{L} the G -linearized invertible sheaf on $Y = G/H$ associated with the H -module L (as in [Ja03, I.5.8]), we then have

$$\mathcal{O}(X) \cong \bigoplus_{i=0}^{p-1} (\mathcal{O}(G) \otimes L^{-i})^H \cong \bigoplus_{i=0}^{p-1} \Gamma(Y, \mathcal{L}^{-i}).$$

Assume in addition that Y is smooth, anti-affine and non-trivial. Then the section s identifies Y to the largest smooth subscheme of X . It remains to check that X is anti-affine; for this, we show that $\Gamma(Y, \mathcal{L}^{-i}) = 0$ for all $i \geq 1$. Consider $\Gamma(Y, \mathcal{L}) = (\mathcal{O}(G) \otimes L)^H$. The exact sequence of H -modules $0 \rightarrow N \rightarrow M \rightarrow L \rightarrow 0$ yields a morphism of G -modules $(\mathcal{O}(G) \otimes M)^H \rightarrow \Gamma(Y, \mathcal{L})$. Moreover, we have an isomorphism of G -modules $(\mathcal{O}(G) \otimes M)^H \cong \mathcal{O}(G/H) \otimes M = M$ in view of [Ja03, I.3.6]. This defines a morphism of G -modules $\varphi : M \rightarrow \Gamma(Y, \mathcal{L})$, dual to the immersion of Y into the projective space of hyperplanes in M . In particular, $\varphi(N)$ is non-zero and consists of sections $\sigma \in \Gamma(Y, \mathcal{L})$ that vanish at the base point y , i.e., $\sigma_y \in \mathfrak{m}_y \mathcal{L}_y$. Choose such a section $\sigma \neq 0$ and let $\tau \in \Gamma(Y, \mathcal{L}^{-i})$. Then $\sigma^i \tau \in \Gamma(Y, \mathcal{O}_Y) = k$ and $\sigma^i \tau$ vanishes at y as well. Thus, $\sigma^i \tau = 0$, hence $\tau = 0$ as Y is smooth and geometrically irreducible.

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